A STUDY OF THE CONTINUOUS AND DISCRETE GAMBIER SYSTEMS

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Abstract

We present a systematic study of the Gambier system, which in the continuous case is given by two Riccati equations in cascade. We derive the condition for its integrability and show that the generic Gambier system contains one free function. We also derive the Schlesinger transformations for this system which allows in principle the systematic construction of the integrable cases. The above procedure is carried over to a discrete setting. We show thus how the discrete Gambier system can be expressed as a system of two homographic mappings in cascade. The integrable cases are obtained through the singularity confinement discrete integrability criterion. Finally the discrete Schlesinger transformations are also derived giving a handle to the construction of the integrable Gambier mapping.

1. Introduction

Linearisable equations play a very particular role in the domain of nonlinear systems. Their nonlinearity is not a genuine one in the sense that elementary transformations can reduce them to a linear system. Calogero coined the term C-integrability (which applies to both ordinary and partial differential equations) in order to describe this situation and to distinguish it from the more complicated situations of systems integrable through IST methods (S-integrability). The simplest example of a linearisable system is the Riccati equation [1]:

$$x' = ax^2 + bx + c \tag{1.1}$$

which can be transformed through the Cole-Hopf transformation $x = -u'/(au)$ to the linear second-order equation:

$$u'' = (b + a'/a)u' - cau \tag{1.2}$$

Higher degree, first-order linearisable equations also exist [1]. Already limiting ourselves to equations of binomial type we find

$$(x')^2 = (x-a)^2(x-k_1)(x-k_2) \tag{1.3}$$

where a is a function of the independent variable and k_1, k_2 are constants. Putting $u^2 = (x-k_1)/(x-k_2)$ equation (1.3) reduces to the Riccati:

$$u' = \pm\frac{1}{2}(k_1 - a - (k_2 - a)u^2) \tag{1.4}$$

When we consider second-order equations, the situation becomes immediately richer, in the sense that there exist several types of linearisable equation [1]. The first (and simplest) one is the equation obtained if one computes the derivative of both sides of the Riccati (1.1). It is usually given in canonical form and reads:

$$x'' = -2xx' + bx' + b'x \tag{1.5}$$

In the same spirit, one can compute the derivative of (1.1) after having divided both sides of the equation by x. We find thus the equation (again given in canonical form):

$$x'' = \frac{x'^2}{x} + \left(ax - \frac{c}{x}\right)x' + a'x^2 + c' \tag{1.6}$$

Another linearisable equation does exist which can be linearized by a Cole-Hopf transformation just like the Riccati. Its form is:

$$x'' = -3xx' - x^3 + q(x' + x^2) \tag{1.7}$$

Indeed, putting $x = u'/u$ we can reduce it to the linear third order equation:

$$u''' = qu'' \tag{1.8}$$

But the most interesting linearisable equation discovered at second-order is the one obtained by Gambier in his classification of second-order equations having the Painlevé property [2]. The Gambier equation is usually given as a second-order equation for a single variable:

$$x'' = \frac{n-1}{n}\frac{x'^2}{x} + a\frac{n+2}{n}xx' + bx' - \frac{n-2}{n}\frac{x'}{x}\sigma - \frac{a^2}{n}x^3 + (a' - ab)x^2 \\ + \Big(cn - \frac{2a\sigma}{n}\Big)x - b\sigma - \frac{\sigma^2}{nx} \tag{1.9}$$

where n is integer, a, b, c are functions of the independent variable and σ is equal to 0 or 1. Its structure becomes clearer when one writes it as a system. We have in this case a system of two Riccati's:

$$y' = -y^2 + by + c \tag{1.10a}$$

$$x' = ax^2 + nxy + \sigma \tag{1.10b}$$

Although the Gambier equation is always linearisable this does not mean it is always integrable. Indeed, in a system such as (1.10) of two equations in cascade we can always solve the first equation for y, obtain $y(t)$ and inject it into the second equation. The latter can always be written as a linear, second-order differential equation for x. So, in principle, the problem can always be solved formally. The difficulty comes when one wishes to actually compute x, in terms of contour integrals, while y has an analytic structure that interferes badly with that of x. This is where the Painlevé property comes into play. If we require that the system possess the Painlevé property the integration can be performed and we can indeed obtain the solution for $x(t)$ over the complex t-plane. Thus, the integrability of the Gambier equation will be closely related to its singularity structure.

The fact that the integrability of the Gambier equation is related to the Painlevé property allows us to obtain another interesting result. Just as in the case of the Painlevé equations, it is possible to introduce transformations relating the solution of an equation with parameter n to one with parameter $n+1$ or $n+2$ (two different transformations do exist). Thus the Gambier system possesses Schlesinger transformations [3].

Another feature of the Gambier equation is that it can be integrably discretized [4,8]. Our approach follows closely the spirit of Gambier based on coupled Riccati equations. In perfect analogy to the continuous case it is possible to introduce a system of two homographic mappings in cascade which represent the discretization of the Gambier system. The general form of this system is the following:

$$y_{n+1} = \frac{ay_n + b}{cy_n + d} \tag{1.11a}$$

$$x_{n+1} = \frac{(\alpha y_n + \beta)x_n + \gamma y_n + \delta}{(\epsilon y_n + \zeta)x_n + \eta y_n + \theta} \tag{1.11b}$$

The form (1.11) can be simplified through homographic transformations and the integrable cases can be obtained through the application of the singularity confinement

discrete integrability criterion. As in the continuous case, although (1.11) is always linearisable it is not automatically integrable. The difficulty arises when one tries to compute x_n in terms of matrix products the elements of which contain y_n. Some of these matrices are singular in such a way that degrees of freedom are irretrievably lost when y_n has the wrong properties. Singularity confinement precisely means that these degrees of freedom are in fact recovered at some later stage.

Quite expectedly, the Gambier mapping has Schlesinger transformations. Their derivation is based on a study of the singularities of the mapping (1.11) and their confinement. Thus the parallel between the discrete and continuous cases is perfect.

In what follows, we shall present the singularity analysis of the Gambier equation and derive its integrable cases. Next, we study the discrete case and use the singularity structure in order to derive the Gambier mapping. Finally, we present the Schlesinger transformations of both the continuous and discrete Gambier systems which allow us, starting from some elementary case (n=0 or n=1), to construct recursively the Gambier systems for higher n's.

2. The continuous Gambier equation

The Gambier equation is given as a system of two Riccati equations in cascade. This means that we start with a first Riccati for some variable y

$$y' = -y^2 + by + c \tag{2.1}$$

and then couple its solution to a second Riccati by making the coefficients of the latter depend explicitly on y:

$$x' = ax^2 + nxy + \sigma. \tag{2.2}$$

The precise form of the coupling introduced in (2.2) is due to integrability requirements. In fact, the application of singularity analysis shows that the Gambier system cannot be integrable unless the coefficient of the xy term in (2.2) is an integer n. This is not the only integrability requirement. Depending on the value of n one can find constraints on the a, b, c, σ (where the latter is traditionnally taken to be constant 1 or 0) which are necessary for integrability.

The common lore [1] is that out of the functions a, b, c two are free. This turns out not to be the case. The reason for this is that the system (2.1-2) is not exactly canonical i.e. we have not used all possible transformations in order to reduce its form. We introduce a change of independent variable from t to T through $dt = gdT$ where g is given by $\frac{1}{g}\frac{dg}{dt} = b\frac{n}{2-n}$, a gauge through $x = gX$ and also $Y = gy - \frac{1}{n}\frac{dg}{dt}$. The net result is that system (2.1-2) reduces to one where $b = 0$ while σ remains equal to 0 or 1. It is clear from the equations above that n must be different from 2. On the other hand when $n = 2$ the integrability conditions, if $\sigma = 1$, is precisely $b = 0$. So we can always take $b = 0$. (As a matter of fact in the case $\sigma = 0$ an additional gauge freedom allows us to take both b and c to zero for all n, even for $n = 2$). Thus the Gambier system can be written in full generality

$$y' = -y^2 + c \tag{2.3a}$$

$$x' = ax^2 + nxy + \sigma. \tag{2.3b}$$

One further remark is in order here. The system (2.3) retains its form under the transformation $x \to 1/x$. In this case $n \to -n$ and σ and $-a$ are exchanged. Thus in some cases it will be interesting to consider a Gambier system where σ is not constant but rather a function of t. Still, it is possible to show that we can always reduce this case to one where $\sigma = 1$, while preserving the form of (2.3a) i.e. $b = 0$. To this end we introduce the change of variables $dt = hdT$, $x = gX$ and $Y = hy - \frac{1}{2}\frac{dh}{dt}$ with $h = \sigma^{2/(n-2)}$, $g = \sigma^{n/(n-2)}$. With these transformations system (2.3) reduces to one with $\sigma = 1$ and $b = 0$. (In the special case $n = 2$, with $b = 0$, integrability implies σ =constant, whereupon its value can always be reduced to 1).

In order to study the movable singularities of the coupled Riccati system we start from the observation that from (2.3a) the dominant behaviour of y can only be $y \approx 1/(t - t_0)$. The next terms in the expansion of y can be easily obtained, and involve the function c and its derivatives. In order to study the structure of the singularities of (2.3b), we first remark that since the latter is a Riccati, its movable singularities are poles. However, (2.3b) also has singularities that are due to the singular behaviour of the coefficients of the r.h.s of (2.3b), namely y. Now, the locations of the singularities of the coefficients are 'fixed' as far as (2.3b) is concerned. However, from the point of view of the full system(2.3), these singularities are *movable* and thus should be studied. The 'fixed' character reflects itself in the fact -1 is *not* a resonance. (The terms 'resonance' is used here following the ARS terminology [6] and means the order, in the expansion, where a free coefficient enters. A resonance -1 is related to the arbitrariness of the location of the singularity, and is thus absent when the location of the singularity is determined from the 'outside' rather than by the initial conditions). Because of the pole in y, x has a singular expansion with a resonance different from -1 which may introduce a compatibility condition to be satisfied.

We consider below the case of the full Riccati (2.3b) with $a \neq 0$, $\sigma = 1$. (The analysis of the case $a\sigma = 0$ was given in [4]). As we explained above, only the singularity due to y can lead to trouble. Rewriting (2.3b) as $x'/x = ax + ny + \sigma/x$ for $y = 1/(t - t_0) + \ldots$ we remark that unless $n = \pm 1$ a behaviour of the form $x \sim (t - t_0)^n$ is impossible when $a\sigma \neq 0$. For $n = 1$, a logarithmic leading behaviour will be present for $\sigma \neq 0$. (Note that the condition $\sigma = 0$ is sufficient for the absence of a critical singularity for $n = 1$ irrespective of the value of a. Similarly, in a dual way, for $n = -1$ the necessary and sufficient condition for the absence of a critical singularity is $a = 0$, whether σ is 1 or 0).

Next we assume $n \neq \pm 1$, in which case it suffices to study the singularities $x \approx \lambda(t - t_0)$, $(\lambda = \sigma/(1 - n))$ and $x \approx \mu/(t - t_0)$, $(\mu = -(n + 1)/a)$. The first singular behaviour ($x \approx \lambda(t - t_0)$) has a resonance at $n - 1$, which is negative for $n < 1$ and thus does not introduce any further condition. For $n > 1$, the resonance condition can be studied at least for the first few values of n. (In fact, $\sigma = 0$ suffices for the resonance condition to be satisfied even for $a \neq 0$). In the particular case $n = 2$, we have already

mentioned that the integrability condition for $\sigma = 1$ is precisely $b = 0$. For higher n (and $\sigma = 1$) we find the further possibilities:

$$n = 3 \qquad 2c - a = 0 \tag{2.5}$$

$$n = 4 \qquad 3c' - a' = 0$$

The second singular behaviour, $x \approx \mu/(t - t_0)$, has a resonance at $-1 - n$. Thus for $n > 0$ this resonance is negative and does not introduce any further condition, while for $n < 0$ a compatibility condition must be satisfied. (We find that for every case $n < 0$, $a = 0$ is a sufficient condition for the absence of critical singularity. This is not in the least astonishing given the duality of a and σ). On the other hand if we demand $a \neq 0$ then a different resonance condition is obtained, at each value of n. For $n = -1$, whenever $a \neq 0$, a logarithmic singularity of the form $(t - t_0)^{-1} \log(t - t_0)$ appears irrespective of the value of σ. For $n < -1$, we find:

$$n = -2 \quad a' = 0$$

$$n = -3 \quad 2ac - a^2\sigma - 2a'' = 0 \tag{2.6}$$

$$n = -4 \quad 2ac' + a'(4c - 2a\sigma) - a''' = 0$$

The integrability condition for higher values of n can be obtained through the use of computer algebra.

3. The discrete Gambier equation

The discretisation of the Gambier equation is based on the idea of two Riccati equations in cascade. The discrete form of the first is simply:

$$\overline{y} = \frac{ay + b}{cy + d} \tag{3.1}$$

where $y \equiv y_n$ and $\overline{y} \equiv y_{n+1}$. The second equation which contains the coupling can be discretised in several, not necessarily equivalent, ways. In [5] we have considered the generic coupling of the form:

$$\alpha x\overline{x}y + \beta x\overline{x} + \gamma\overline{x}y + \delta\overline{x} + \epsilon xy + \zeta x + \eta y + \theta = 0 \tag{3.2}$$

Implementing a homographic transformation on x and y we can generically bring (3.2) under the form:

$$x\overline{x} + \gamma\overline{x}y - \epsilon xy - \theta = 0 \tag{3.3}$$

. A choice of different transformations can bring (3.2) also to the form

$$\overline{x} - x = -fx\overline{x} + (gx + h\overline{x})y + k \tag{3.4}$$

Note that (3.2) contains an ‘additive’ type coupling $x\overline{x}+\delta\overline{x}+\zeta x+\eta y+\theta=0$ for special values of its parameters, but the generic form (3.3) is that of a ‘multiplicative’ coupling where γ, ϵ do not vanish. Solving (3.3) for $\overline{x}$ we obtained the second equation of the discrete Gambier system in the form:

$$\overline{x}=\frac{\epsilon xy+\theta}{x+\gamma y}. \tag{3.5}$$

Clearly, a scaling freedom remains in equation (3.5). We can use it in order to bring it to the final form:

$$\overline{x}=\frac{xy/d+c^2}{x+dy} \tag{3.6}$$

Eliminating y and $\overline{y}$ from (3.1), (3.6) and its upshift, we can obtain a 3-point mapping for x alone but the analysis is clearer if we deal with both y and x.

In what follows we shall present a different derivation based on the study of singularities of the system. As we explained in [3] we can use the homographic freedom in order to bring the mapping for y to the form:

$$\overline{y}=\frac{y+c}{y+1} \tag{3.7}$$

instead of (3.1) where c is a function of n. Next, we turn to the equation for x. This equation is homographic in x. However we require that when y takes the value 0, the resulting value of x be ∞. Thus the denominator must be proportional to y, and since we can freely translate x, we can reduce its form to just xy. The remaining overall gauge factor is chosen so as to put the coefficient of xy of the numerator to unity resulting to the following mapping:

$$\overline{x}=\frac{x(y-r)+q(y-s)}{xy}. \tag{3.8}$$

The system (3.7-8) is a discrete form of the Gambier system. In order to study the confinement of the singularity induced by $y=0$ we introduce the auxiliary quantity ψ_N which is the N’th iterate of $y=0$ in equation (3.7), N times downshifted. Thus $\psi_0=0$, $\psi_1=\underline{c}$, $\psi_2=\frac{\underline{\underline{c}}+\underline{c}}{\underline{\underline{c}}+1}$, etc... The confinement requirement is that after N steps x becomes 0 in such a way as to lead to $0/0$ at the next step. Thus the mapping (3.8) has in fact the form:

$$\overline{x}=\frac{x(y-r)+q(y-\psi_N)}{xy}. \tag{3.9}$$

Thus when at some step N we have $y=\psi_N$ and $x=0$, on the view of (3.9) $\overline{x}$ will then be indeterminate of the form $0/0$. However it turns out that in fact this value is well-determined and finite. Let us take a closer look at the conditions for confinement. The generic patterns for x and y are:

$$\begin{array}{lcccccccc} y:\{ & & 0 & \overline{\psi_1} & & \overline{\overline{\psi_2}} & \dots & \overline{\overline{\overset{\vdots}{\psi_N}}} & & \} \\ x:\{ & \text{free} & \infty & & \dfrac{\overline{\psi_1}-\overline{r}}{\overline{\overline{\psi_1}}} & & \dots & 0 & \text{free} & \}. \end{array}$$

At $N=1$ it is clearly impossible to confine with a form (3.9) since we do not have enough steps. In this case the only integrable form of the x-equation is a linear one. This case must be studed separately. For an arbitrary N, the general form of the linear x-mapping can be obtained using confinement arguments in a way similar to what we did for the generic, nonlinear, case. We obtain

$$\overline{x} = \frac{x(y-\psi_N)+g}{y} \tag{3.10}$$

where g is free.

The first genuinely confining case of the form (3.9) is $N=2$. From the requirement $\overline{\overline{x}}=0$ we have $r=\psi_1$ and q free: this is indeed the only integrability condition. For higher N's we can similarly obtain the confinement condition which takes the form of an equation for r in terms of q.

At this point it is natural to ask whether the mapping (3.7)-(3.9) does indeed correspond to the Gambier equation (2.3). In order to do this we construct its continuous limit. We first introduce:

$$\begin{gathered} c=\epsilon^2 D \\ y=\frac{\epsilon D}{Y+H} \end{gathered} \tag{3.11}$$

with $H \approx D'/(2D)$ and obtain the continuous limit of (3.7) for $\epsilon \to 0$. We find as expected

$$Y' = -Y^2 + C \tag{3.12}$$

(i.e. eq. (2.3a)) where $C = D - \frac{D''}{2D} + \frac{3}{4}\frac{D'^2}{D^2}$. Using (3.7) and (3.11) we can also compute ψ_N and we find at lowest order:

$$\psi_N = \epsilon^2 \Psi_N \qquad \text{with} \qquad \Psi_N \approx N(D - \epsilon\frac{N+1}{2}D') + \epsilon^2 \Phi_N \tag{3.13}$$

where Φ_N is an explicit function of D depending on N.

Next we turn to the equation for x and introduce:

$$\begin{gathered} r=\epsilon^2 R \\ x=\frac{1}{2}+\frac{\epsilon}{2X}-\epsilon\frac{RD'}{4D^2} \\ q \approx -\frac{1}{4}+\epsilon^2 Q \end{gathered} \tag{3.14}$$

and for the continuous limit of the form (2.3b) to exist in canonical form (i.e. b=0, σ=1) we find that we must have

$$R \approx \frac{ND}{2} - \epsilon(N+2)\frac{ND'}{8}. \tag{3.15}$$

This leads to the equation for x:

$$X' = AX^2 + NXY + 1 \tag{3.16}$$

with $A = \frac{N}{4}(N/4+1)\frac{D'^2}{D^2} - \frac{ND''}{4D} - 4Q$. Moreover the confinement constraint implies a differential relation between D and Q which depends on N. We can verify explicitly in the first few cases that this is indeed the integrability constraint obtained in the continuous case. In the case $N=2$, the condition for confinement is

$$\overline{r} = c. \tag{3.17}$$

Using the approximate expansion in ϵ (3.15) which gives r up to the third order in ϵ and putting $N=2$, we get that equation (3.17) is automaticaly satisfied for the two first nonzero orders in ϵ (ϵ^2 and ϵ^3). This reflects the fact that for the continuous Gambier equation, in the case $N=2$, the only integrability condition is $b=0$. Thus no condition has to be imposed on D and Q.

The confinement condition for $N=3$ is obtained by imposing $\overline{\overline{\overline{x}}} = 0$ when $y=0$. This condition reads:

$$(\overline{c} + c - (\overline{c}+1)\overline{\overline{\psi_3}}) = (\overline{r} - c)(\overline{c} + c - \overline{\overline{r}}(\overline{c}+1)). \tag{3.18}$$

Putting $N=3$, we find that equation (3.18) is identically satisfied for the first two nonzero orders in ϵ (ϵ^4 and ϵ^5). To get the integrability condition, we must calculate (3.18) at order ϵ^6. To do this, we need the value of Φ_3 in (3.13). We easily find that $\Phi_3 = 7D'' - 8D^2$. Satisfying (3.18) in ϵ^5 gives a relation giving explicitly Q in terms of D. Implementing this relation, we find that $A = 2C$ which is the continuous integrability condition for $N=3$.

4. Schlesinger transformations for the continuous Gambier equation

The theory of auto-Bäcklund transformations of Painlevé equations is well established. As was shown in [7] the general form of auto-Bäcklund transformations for most Painlevé equations is of the form:

$$\tilde{x} = \frac{\alpha x' + \beta x^2 + \gamma x + \delta}{\epsilon x' + \zeta x^2 + \eta x + \theta}. \tag{4.1}$$

In the case of the Gambier equation considered as a coupled system of two Riccati's it is more convenient to look for an auto-Bäcklund of the form:

$$\tilde{x} = \frac{\alpha xy + \beta x + \gamma y + \delta}{(\zeta y + \eta)(\theta x + \kappa)}. \tag{4.2}$$

with a factorized denominator, with hindsight from the discrete case. We require that the equation satisfied by $\tilde{x}$ do not comprise terms nonlinear in y. We examine first the case $\zeta \neq 0$ and reach easily the conclusion that there exists no solution. So we take $\zeta = 0$, $\eta = 1$ which implies that α and γ do not both vanish (otherwise (4.2) would have been independent of y). We find in this case $\alpha = 0$ and thus the general form of the auto-Bäcklund can be written as:

$$\tilde{x} = \frac{\beta x + \gamma y + \delta}{\theta x + \kappa}. \tag{4.3}$$

From (4.3) we can obtain the two possible forms of the Gambier system auto-Bäcklund:

$$\tilde{x} = \beta x + \gamma y + \delta \tag{4.4}$$

$$\tilde{x} = \frac{\beta x + \gamma y + \delta}{x + \kappa}. \tag{4.5}$$

As we shall see in what follows both forms lead to Schlesinger transformations.

Let us first work with form (4.4). Our approach is straightforward. We assume (4.4) and require that $\tilde{x}$ satisfy an equation of the form (2.3b) while y is always the same solution of (2.3a). The calculation is easily performed leading to:

$$\tilde{x} = \gamma y + \frac{a\gamma}{n+1}x + \frac{\gamma'}{n}, \tag{4.6}$$

where γ satisfies:

$$\frac{\gamma'}{\gamma} = \frac{n}{n+2}\frac{a'}{a}. \tag{4.7}$$

Here we have assumed $a \neq 0$; otherwise $\tilde{x}$ does not depend on x and (4.6) does not define a Schlesinger. The parameters of the equation satisfied by $\tilde{x}$ are given (in obvious notations) by:

$$\tilde{n} + n + 2 = 0 \tag{4.8a}$$

$$\tilde{a} = \frac{n+1}{\gamma} \tag{4.8b}$$

and

$$\tilde{\sigma} = \gamma\Big(c + \frac{a\sigma}{n+1} + \frac{1}{n+2}\frac{a''}{a} - \frac{n+3}{(n+2)^2}\frac{a'^2}{a^2}\Big). \tag{4.8c}$$

Thus (4.6) is indeed a Schlesinger transformation since it takes us from a Gambier system with parameter n to one with parameter $\tilde{n} = -n - 2$. It suffices now to invert $\tilde{x}$ in order to obtain an equation with parameter $N = n + 2$. Expressions (4.6) and (4.8) can be written in a more symmetric way:

$$\tilde{a}\tilde{x} - ax = (n+1)(y - \frac{a'}{\tilde{n}a}) \tag{4.9}$$

and

$$\begin{gathered} \tilde{n} + 1 = -(n+1) \\ \tilde{n}\frac{\tilde{a}'}{\tilde{a}} = n\frac{a'}{a} \\ \tilde{a}\tilde{\sigma} - a\sigma = (n+1)\Big(c - \frac{1}{\tilde{n}}\big(\frac{a'}{a}\big)' + \frac{1}{\tilde{n}^2}\frac{a'^2}{a^2}\Big). \end{gathered} \tag{4.10}$$

The inverse transformation can be easily obtained if we introduce $\tilde{\gamma}$ such that $a\tilde{\gamma} = -(n+1) = -\tilde{a}\gamma$. We thus find

$$x = y\tilde{\gamma} + \frac{\tilde{a}\tilde{\gamma}}{\tilde{n}+1}\tilde{x} + \frac{\tilde{\gamma}'}{\tilde{n}} \tag{4.11}$$

and the relations (4.10) are still valid.

Iterating the Schlesinger transformations one can construct the integrable Gambier systems for higher n's and obtain by construction the functions which appear in them. However it may happen that when we implement the Schlesinger we find $\tilde{\sigma} = 0$. If we invert x we get a system with $N = -\tilde{n} = n + 2$ but $A = 0$ for which one cannot iterate the Schlesinger.

Let us give example of the application of this Schlesinger transformation. Let us start from $n = 0$, in which case we find $\tilde{n} = -2$ and, after inversion, $N = 2$. For $n = 0$ we start from $a = -1$ and $\sigma = 0$ or 1 (always possible through the appropriate changes of variable). This leads to $\tilde{a} = -1$, $\tilde{\sigma} = -c + \sigma$ and the Schlesinger reads: $\tilde{x} = -y + x$. Next we invert $\tilde{x}$ and have $X = 1/(x - y)$. We find thus that the Schlesinger takes us from

$$\begin{aligned} y' &= -y^2 + c \\ x' &= -x^2 + \sigma \end{aligned} \tag{4.12}$$

to the system

$$\begin{aligned} y' &= -y^2 + c \\ X' &= AX^2 + 2Xy + \Sigma \end{aligned} \tag{4.13}$$

with $A = c - \sigma$, $\Sigma = 1$. In the particular case $n = 2$, a change of variables exists which allows us to put $A = -1$ (unless $A = 0$), without introducing b in the equation for y, while keeping $\Sigma = 1$ and changing only the value of c. Thus the generic case of the Gambier equation for $n = 2$ can be written with $A = -1$. Eliminating y between the two equations we find:

$$x'' = \frac{x'^2}{2x} - 2xx' - \frac{x^3}{2} - \frac{1}{2x} + (2c + 1)x. \tag{4.14}$$

This is the generic form of the $n = 2$, Gambier equation [1] and it contains just one free function.

We turn now to the second Schlesinger transformation corresponding to the form (4.5). As we shall show, a Schlesinger transformation of this form does indeed exist and corresponds to changes in n with $\Delta n = 1$. Let us start from the basic equations (2.3). Next we ask that $\tilde{x}$ defined by (4.5) indeed satisfy a system like (2.3). We find thus that and $\kappa = -x_0$ and γ must be given by:

$$\frac{\gamma'}{\gamma} = y_0 + \frac{2ax_0}{n + 1} \tag{4.15}$$

where y_0 is a solution of the Riccati (2.3a) and a solution x_0 of (2.3b), obtained with y replaced by y_0. We introduce the quantities $\tilde{x}_0 = \frac{a\gamma}{n+1}$, $\tilde{a} = -\frac{nx_0}{\gamma}$. In this case (4.15) becomes:

$$\frac{\gamma'}{\gamma} = y_0 + \frac{2\tilde{a}\tilde{x}_0}{\tilde{n} + 1} = y_0 + \frac{2x_0\tilde{x}_0}{\gamma}, \tag{4.16}$$

where

$$\tilde{n} + n + 1 = 0. \tag{4.17}$$

We have thus, starting from a generic solution x, y of (2.3) for some n, the Schlesinger:

$$\tilde{x} = \tilde{x}_0 + \frac{\gamma(y - y_0)}{x - x_0} \tag{4.18}$$

where $\tilde{x}$ is indeed a solution of (2.3) for $\tilde{n} = -n-1$ for the *same* y

$$\tilde{x}' = \tilde{a}\tilde{x}^2 + \tilde{n}\tilde{x}y + \tilde{\sigma} \tag{4.19}$$

where $\tilde{a}$ has been defined as $-nx_0/\gamma$ and

$$\tilde{\sigma} = \frac{\gamma}{n+1}\Big(a' + a^2 x_0 \frac{n+2}{n+1} + a y_0 (n+2)\Big). \tag{4.20}$$

Note that $\tilde{x}_0$ is a solution of the same equation with y replaced by y_0. As in the previous case if we invert $\tilde{x}$ we obtain an equation corresponding to $N = n+1$.

It is worth pointing out here that the Schlesinger transformation corresponding to $\Delta n = 2$ was known to Gambier himself. As a matter of fact when faced with the problem of determining the functions appearing in his equation so as to satisfy the integrability requirement, Gambier proposed a recursive method which is essentially the Schlesinger $\Delta n = 2$. On the other hand the Schlesinger $\Delta n = 1$ is quite new.

5. Schlesinger transformations for the discrete Gambier equation

Once the singularity pattern of the Gambier mapping is established we can use it in order to construct the Schlesinger transformation. Let us first look for a transformation that corresponds to $\Delta N = 2$. The idea is that given the N-steps singularity pattern of the equation for x we introduce a variable w with $N+2$ singularity steps where we enter the singularity one step before x and exit it one step later. The general form of the Schlesinger transformation, which defines w, is:

$$w = X\frac{y - \psi_{N+1}}{y}, \tag{5.1}$$

where X is homographic in x. The presence of the y and $y - \psi_{N+1}$ terms is clear: they ensure that w becomes infinite one step before x, and vanishes one step after x. Next we turn to the determination of X. Since X is homographic in x we can rewrite (5.1) as:

$$w = \frac{\alpha x + \beta}{y}\frac{y - \psi_{N+1}}{\gamma x + \delta}. \tag{5.2}$$

Our requirement is that w becomes infinite when $y = 0$ for every value of x. This statement must be qualified. The numerator $\alpha x + \beta$ *will* vanish for some x (namely $x = -\beta/\alpha$) so this value of x must be the only one which should *not* occur in the confined singularity. Indeed there is a unique value of x where instead of being confined, the singularity extends to infinity in *both* directions of the independent variable n, while the only nonsingular values of the dependent variable occur in a finite range. The value of x such that $\overline{x}$ is finite and free even though y is zero is such that the numerator $-xr - q\psi_N$ of $\overline{x}$ vanishes. For this value of x, the values of the dependent variable are fixed for $n \leq 0$ and $n \geq N+1$ and the value can be considered as 'forbidden'. Thus $\alpha x + \beta = xr + q\psi_N$ up to a multiplicative constant. Similarly when $y = \psi_{N+1}$, w must vanish. Thus $\gamma x + \delta$ must not be zero except for the unique value of x that does not

occur in the confined singularity. Note that $y=\psi_{N+1}$ means $\underline{y}=\underline{\psi}_N$ and the only value of x that comes from a nonzero $\underline{x}$ in that case is $x=(\underline{\psi}_N-\underline{r})/\underline{\psi}_N$. In that case the values of the dependent variable are fixed for $n\geq 0$ and $n\leq -N-1$. This value of x being 'forbidden', $\gamma x+\delta$ must be proportional to $\underline{\psi}_N x-(\underline{\psi}_N-\underline{r})$. We now have the first form of the Schlesinger:

$$w=\frac{xr+q\psi_N}{y}\frac{y-\psi_{N+1}}{\underline{\psi}_N x+\underline{r}-\underline{\psi}_N} \tag{5.3}$$

where the proportionality constant has been taken equal to 1 (but any other value would have been equally acceptable). Here w effectively depends on x unless $r(\underline{r}-\underline{\psi}_N)=q\underline{\psi}_N\psi_N$. But in this case the mapping (3.8) is in fact linear in the variable $\xi=(x-1+\overline{r}/\overline{\psi}_N)^{-1}$. This case is the analog of the case $a=0$ in the continuous case where the Schlesinger does not exist. Let us give an application of the Schlesinger transformation by obtaining the $N=2$ equation starting from $N=0$. We have always the equation for y which reads:

$$\overline{y}=\frac{y+c}{y+1} \tag{5.4}$$

and $\psi_0=0$, $\psi_1=\underline{c}$. For $N=0$ the equation for x reads:

$$\overline{x}=\frac{x(y-r)+qy}{xy}=\frac{x+q}{x} \tag{5.5}$$

since for integrability $r=0$ and indeed $N=0$ means that the x equation does *not* depend on y. We introduce the Schlesinger:

$$w=x\frac{y-\underline{c}}{y} \tag{5.6}$$

Using (5.5) and (5.6) to eliminate x we obtain the equation for w:

$$\overline{w}=(1-c)\frac{yw+q(y-\underline{c})}{(y+c)w}. \tag{5.7}$$

This equation is of the form (3.8) but not quite canonical. We can transform it to canonical form simply by introducing $\overline{y}$ instead of y because indeed w is infinite one step before x, so $w=\infty$ means $\overline{x}=\infty$ i.e. $\overline{y}=0$. We obtain thus:

$$\overline{w}=\frac{w(\overline{y}-c)+q(1+\underline{c})(\overline{y}-\overline{\psi}_2)}{\overline{y}w} \tag{5.8}$$

with $\overline{\psi}_2=(c+\underline{c})/(1+\underline{c})$ which coupled to (5.4) is indeed a $N=2$ Gambier mapping.

As we pointed out in section 3, necessitates a special treatment. The Schlesinger transformation is again given by:

$$w=X\frac{y-\psi_{N+1}}{y} \tag{5.9}$$

and arguments similar to those of the nonlinear case allow us to determine the form of the homographic object X leading to:

$$w = \frac{x\psi_N - g}{y} \frac{y - \psi_{N+1}}{x\underline{\psi}_N - \underline{g}}. \tag{5.10}$$

Thus one can also perform a Schlesinger in the linear case. This is not in disagrement with the continuous case. It is, in fact, the analog of the case where $\sigma = 0$ but $a \neq 0$ (which is linear in $1/x$) for which the Schlesinger *can* be performed. The analog of the case $\sigma = 0$ and $a = 0$ is the situation when $g = k\psi_N$ with constant k in which case the mapping rewrites $\overline{\xi} = \xi(y - \psi_N)/y$ with $\xi = x - k$. Then w does not depend on ξ (or x) and (4.20) does not define a Schlesinger in analogy to the case $r(\underline{r} - \underline{\psi}_N) = q\underline{\psi}_N\psi_N$ in the nonlinear case.

Finally, we examine the possibility of the existence of a $\Delta N = 1$ Schlesinger. In this case, the structure of the transformation will be obtained by asking that the $N+1$ case enter the singularity one step before the N case but exit at the same point. The general structure is thus:

$$w = \frac{rx + q\psi_N}{y} \frac{y - \eta}{x - \xi} \tag{5.11}$$

where η and ξ must be determined. We do this by requiring that the equation for w contain no coefficients nonlinear in y. As a result we find that η must satisfy the equation (3.7) for y:

$$\overline{\eta} = \frac{\eta + c}{\eta + 1} \tag{5.12}$$

and ξ the equation (3.9) for x with η instead of y:

$$\overline{\xi} = \frac{\xi(\eta - r) + q(\eta - \psi_N)}{\xi\eta}. \tag{5.13}$$

We remark here the perfect parallel to the continuous case (and as we pointed out the discrete case led the investigation back to the continuous one). Let us point out here that the w obtained through (4.26) does not lead to $w = 0$ at the exit of the singularity (i.e. when $x = 0$, $y = \psi_N$) and a translation is needed. One has in principle to define a new variable

$$\omega = w - w(x = 0, y = \psi_N) = w + \frac{q}{\xi}(\psi_N - \eta).$$

Finally we derive the $\Delta N = 1$ Schlesinger for the case of a linear mapping (5.9). We start from:

$$w = \frac{x\psi_N - g}{y} \frac{y - \eta}{x - \xi} \tag{5.14}$$

and again require for w an equation with coefficients linear in y. We find that η must again be a solution of the equation for y i.e. it must satisfy (5.12) and moreover ξ is a solution of (5.13) with $y = \eta$:

$$\overline{\xi} = \frac{\xi(\eta - \psi_N) + g}{\eta}. \tag{5.15}$$

Thus the list of the Schlesinger transformations of the Gambier mapping is complete.

6. Conclusion

In this work we have reviewed the Gambier system in both its continuous and discrete forms. We have shown that the singularity analysis can be used in order to obtain the integrable subcases of this equation and moreover we have derived their Schlesinger transformations. Several open questions appear at this point. The Gambier equation is the generic linearisable second-order equation of first degree. If one relaxes this last constraint, one can already obtain further linearisable equations. Cosgrove and Scoufis [8] have presented two such examples:

$$(x'')^2 = (ax' + a'x + c')^2 x' \tag{6.1}$$

$$(x'')^2 = A(x)(c_1(tx' - x) + c_2 x'(tx' - x) + c_3(x')^2 + c_4(tx' - x) + c_5 x' + c_6). \tag{6.2}$$

It would be interesting to study closely the properties of these equations and in particular their discretization.

Moving to higher orders one can wonder how the Gambier approach can be extended. In [9] we have presented a first exploration of this problem, in both continuous and discrete settings, for third-order systems.

Finally, the general problem of n-th-order linearizable systems is far from being solved. Only a special class of such systems is known, based on projective contructions. Recently, we have obtained the discretization of these projective Riccati systems [10]. Clearly, this is not the last word as far as linearisability is concerned.

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